Singular Value Decomposition (SVD)
A Fast Track Tutorial

Abstract – This fast track tutorial provides instructions for decomposing a matrix using the singular value decomposition (SVD) algorithm. The tutorial covers singular values, right and left eigenvectors. To complete the proof the full SVD of a matrix is computed.

Keywords: singular value decomposition, SVD, singular values, eigenvectors, full SVD, matrix decomposition
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Background

In 1965, Golub and Kahan (1965) published their famous singular value decomposition (SVD) algorithm for obtaining the decomposition of a given rectangular matrix and its pseudo-inverse.

Their work was first presented at the Symposium on Matrix Computations at Gatlinburg, Tennessee, in April, 1964. A link to their original ground-breaking article is given in the Reference section. The algorithm works as a dimensionality reduction technique and as follows.

1. A rectangular matrix $\mathbf{A}$ is defined and its transpose $\mathbf{A}^{\mathbf{T}}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ product computed.
2. The singular values of $\mathbf{A}$ are obtained by computing the eigenvalues of $\mathbf{A}^{\mathbf{T}}\mathbf{A}$.
3. The diagonal matrix $\mathbf{S}$ and its inverse, $\mathbf{S}^{-1}$, are computed.
4. The eigenvectors of $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ are obtained and $\mathbf{V}$ and its transpose, $\mathbf{V}^{\mathbf{T}}$, computed.
5. $\mathbf{U} = \mathbf{AVS}^{-1}$ is computed. The original matrix can be recovered as $\mathbf{A} = \mathbf{USV}^{\mathbf{T}}$.

An illustrative example follows. We assume that the reader is familiar with basic linear algebra and matrix calculations.

Example

Compute the full SVD for the following matrix.

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Solution

Step 1. Compute its transpose $\mathbf{A^T}$ and $\mathbf{A^TA}$.

Since

$$A^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

then,

$$A^TA = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

Step 2. Determine the eigenvalues of $\mathbf{A^TA}$ and sort these in descending order in the absolute sense. Square roots these to obtain the singular values of **A**.

$$A^TA - cI = \begin{bmatrix} 25 - c & -15 \\ -15 & 25 - c \end{bmatrix}$$

$$| A^TA - cI | = (25 - c)(25 - c) - (-15)(-15) = 0$$

characteristic equation $\longrightarrow$ $c^2 - 50c + 400 = 0$

The quadratic equation gives two values.
In decreasing order, these are $\longrightarrow$ $| 40 | > | 10 |$

eigenvalues $\longrightarrow$ $c_1 = 40 \quad c_2 = 10$

singular values $\longrightarrow$ $s_1 = \sqrt{40} = 6.3245 \quad > \quad s_2 = \sqrt{10} = 3.1622$

Step 3. Construct diagonal matrix **S** by placing singular values in descending order along its diagonal. Compute its inverse, $\mathbf{S^{-1}}$.

$$S = \begin{bmatrix} 6.3245 & 0 \\ 0 & 3.1622 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 0.1581 & 0 \\ 0 & 0.3162 \end{bmatrix}$$

Step 4. Use the ordered eigenvalues from step 2 and compute the eigenvectors of $\mathbf{A^TA}$. Place these eigenvectors along the columns of **V** and compute its transpose, $\mathbf{V^T}$.

for $c_1 = 40$

$$\mathbf{A^TA - cI} = \begin{bmatrix} 25-40 & -15 \\ -15 & 25-40 \end{bmatrix} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$(\mathbf{A^TA - cI})\,\mathbf{X_1} = \mathbf{0}$$

$$\begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-15x_1 + -15x_2 = 0$$
$$-15x_1 + -15x_2 = 0$$

Solving for x_2 for either equation: $x_2 = -x_1$

$$\mathbf{X_1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

Dividing by its length,

$$L = \sqrt{x_1^2 + x_2^2} = x_1\sqrt{2}$$

$$\mathbf{X_1} = \begin{bmatrix} x_1 / L \\ -x_1 / L \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix}$$

for $c_2 = 10$

$$\mathbf{A^TA - cI} = \begin{bmatrix} 25-10 & -15 \\ -15 & 25-10 \end{bmatrix} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$(\mathbf{A^TA - cI})\,\mathbf{X_2} = \mathbf{0}$$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$15x_1 + -15x_2 = 0$$
$$-15x_1 + 15x_2 = 0$$

Solving for x_2 for either equation: $x_2 = x_1$

$$\mathbf{X_2} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

Dividing by its length,

$$L = \sqrt{x_1^2 + x_2^2} = x_1\sqrt{2}$$

$$\mathbf{X_2} = \begin{bmatrix} x_1 / L \\ x_1 / L \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{X_1} & \mathbf{X_2} \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

$$\mathbf{V^T} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

Step 5. Compute $\mathbf{U} = \mathbf{AVS}^{-1}$ as the reduced matrix. To complete the proof, compute the full SVD using $\mathbf{A} = \mathbf{USV}^{\mathbf{T}}$.

$$U = AVS^{-1} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 0.1581 & 0 \\ 0 & 0.3162 \end{bmatrix}$$

$$U = AVS^{-1} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 0.1118 & 0.2236 \\ -0.1118 & 0.2236 \end{bmatrix}$$

$$U = AVS^{-1} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix}$$

$$A = USV^{T} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix} \begin{bmatrix} 6.3245 & 0 \\ 0 & 3.1622 \end{bmatrix} \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

$$A = USV^{T} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix} \begin{bmatrix} 4.4721 & -4.4721 \\ 2.2360 & 2.2360 \end{bmatrix}$$

$$A = USV^{T} = \begin{bmatrix} 3.9998 & 0 \\ 2.9999 & -4.9997 \end{bmatrix} \approx \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

The orthogonal nature of the **V** and **U** matrices is evident by inspecting their eigenvectors. This can be demonstrated by computing dot products between column vectors. All dot products are equal to zero. Alternatively, we can plot these and see that they are orthogonal. See Figure 1.

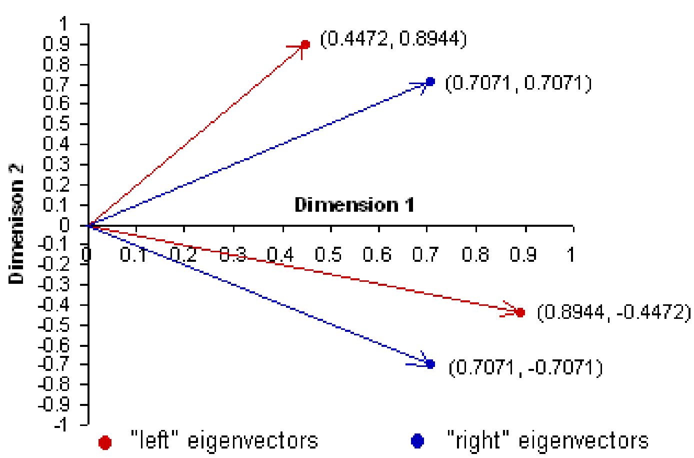

Figure 1. Left and Right Eigenvectors.

Questions

For the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

1. Compute the eigenvalues of $\mathbf{A}^T\mathbf{A}$.
2. Prove that this is a matrix of Rank 2.
3. Compute its full SVD.
4. Compute its Rank 2 Approximation.

Conclusion

In this fast track tutorial we provided instructions for decomposing a matrix using the singular value decomposition (SVD) algorithm. We discussed how singular values and right and left eigenvectors are computed. To complete the proof the full SVD of a matrix was computed.

References

Golub, G., Kahan, W. Calculating the Singular Values and Pseudo-Inverse of a Matrix. J. SIAM Numer. Anal. Ser. B, Vol 2, 2, pp 205-224. Retrieved from http://web.stanford.edu/class/cme324/classics/golub-kahan.pdf